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Existence and Completeness of the Wave Operators for Higher Order Differential Operators

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The aim of the present paper is to study the existence and the completeness of the wave operators $\mathcal{W}_{\pm}(H, H_0)$ for elliptic operators of higher order (Schrödinger operator) with short-range potential of the form:

$$H = \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\alpha|} D^{\alpha}(b_{\alpha\beta}(x)D^{\beta}) + V_0 + V \qquad (*)$$

and other related results by using the trace class method.

KEY WORDS: wave operator; trace class method; partial differential operator.

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1. INTRODUCTION

It is well known that, the spectral information about a selfadjoint operator H can be obtained by using the wave operator. Thus, it is an important problem to find conditions guaranteeing the existence of wave operator. There are different methods used in scattering theory: the smooth method and the trace class method (see Yafaev, 1992, 2000; Birman and Yafaev, 1991). The first of them make essential use of an explicit spectral analysis of the unperturbed operator H_0 while the perturbation $V = H - H_0$ smooth relation to H_0. For example, in the Friedrichs–Faddeev model H_0 acts as multiplication by independent variable in the space $\mathcal{H} = L_2(\Lambda; \mathfrak{H})$ where Λ is an interval and $\mathfrak{H}$ is an auxiliary Hilbert space and the perturbation V is an integral operator with sufficiently smooth kernel (see Faddeev, 1967; Friedrichs, 1948; Yafaev, 1992). The other important type of smooth

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method is defined by H-smoothness which was introduced by Kato (see Kato, 1996). There are several versions of the concept of smoothness of a perturbation. The second method: the trace class method, which is the principle of abstract scattering theory and the trace class theory is different in principle from the smooth theory. The trace class method is based on the fact that for an arbitrary self-adjoint operator H and any operators $\mathcal{G}_1, \mathcal{G}_2$ of the *Hilbert-Schmidt* class $\mathfrak{S}_2$ the product $\mathcal{G}_1 R_0(\lambda \pm i\varepsilon)\mathcal{G}_2$ has a limit in $\mathfrak{S}_2$ as $\varepsilon \to 0$ for almost all $\lambda \in \Re$. The *Kato–Rosenblum* Theorem (Kato, 1957a,b; Rosenblum, 1957) on the existence of the wave operator $\mathcal{W}_{\pm}(H, H_0)$ is the fundamental Theorem for the trace class method. It is shown that the wave operator $\mathcal{W}_{\pm}(H, H_0)$ exist and complete if the difference $H - H_0$ of is a trace class. Trace class method is one of the important methods applicable to the theory of differential operators. For example, the second-order elliptic differential operator in $\Re^3$ was considered by Ikebe and Tayoshi (1968), also by using a trace condition. For some other results connected to the trace condition (see Birman, 1962, 1963, 1964, 1968, 1969, Birman and Entina, 1964). More general, the trace class method is used for perturbations of media. More precise, for the operators $H_0 = M_0^{-1}(x)P(D)$ and $H = M^{-1}(x)P(D)$ where M_0 and $M(x)$ are positively definite bounded matrix-valued functions and $P(D)$ is an elliptic differential operator. The wave operators for the pair H_0 , H exist and are complete if the difference $M(x) - M_0(x) = O(|x|^{-\rho})$, $\rho > d$, as $|x| \longrightarrow \infty$ (see Yafaev, 2004).

The organization of this paper is the following: the results of Section 2, are necessary background for an elementary proof of the existence and completeness of wave operators.

Our goal here to prove the existence and the completeness of the wave operators by using the trace-class method for the Schrödinger operator with magnetic vector potential and for the differential operators of higher order that take the form $(*)$ which shown in Sections 3 and 4 respectively. This generalize the result of Birman and Solomyk (1977) that $H_0 = (-\triangle)^l$, l is an integer.

2. PRELIMINARIES

Let $L^2(\Re^d) = \mathcal{H}$ be the Hilbert space, and let $\mathfrak{B}$ be the algebra of all bounded operators acting on $L^2(\Re^d)$. The ideal of compact operators will be denoted by $\mathfrak{S}_\infty$. Recall that a compact operator T on $\mathcal{H}$ is in the class $\mathfrak{S}_p$, $1 \leq p < \infty$ if

$$||T||_p^p = \sum_{n=1}^{\infty} s_n^p(T) = \sum_{n=1} \lambda_n^p(T^*T)^{1/2} < \infty.$$

where the numbers $s_n(T) = \lambda_n[(T^*T)^{1/2}] = \lambda_n(|T|)$ of any compact operator T is the eigenvalues of the positive compact operator $(T^*T)^{1/2}$ and listed with account of multiplicity in decreasing order which is symmetrically normed ideals of the algebra $\mathfrak{B}$. In particular, $\mathfrak{S}_1$ is called the trace class and $\mathfrak{S}_2$ is called the *Hilbert–Schmidt* class. Clearly, $\mathfrak{S}_{p_1} \subset \mathfrak{S}_{p_2}$ for $p_1 \leq p_2$, and moreover, we have the following:

Lemma 2.1. *If $T \in \mathfrak{S}_r$ and B is a bounded operator, then $TB \in \mathfrak{S}_r$. If $T_j \in \mathfrak{S}_{r_j}$, $j = 1, 2$, then $T_1 T_2 \in \mathfrak{S}_r$, where $r = r_1^{-1} + r_2^{-1}$ (see Birman and Solomyk, 1977).*

Proposition 2.2. *Let*

$$|a_1(x)| \leq c(1 + |x|)^{-\beta}, \quad |a_2(x)| \leq c(1 + |\xi|)^{-\beta}, \quad \beta > 0$$

then the operator T,

$$(Tf)(x) = a_1(x) \int_{E^d} \exp(\pm i < x, \xi >) a_2(\xi) f(\xi) d\xi,$$

belongs to the class $\mathfrak{S}_r$ if $r > d\beta^{-1}$.

For $a_1, a_2 \in L_2(\Re^d)$ we have, obviously, $T \in \mathfrak{S}_2$, while for $T \in \mathfrak{S}_\infty$ it is sufficient that the functions a_1, a_2 be bounded and should tend to zero at infinity (see Birman and Solomyk, 1977).

Proposition 2.3. *Let (M, μ) be a measure space and let T be an integral operator on $L^2(M; d\mu)$ with measurable integral kernel $K(x; y)$. Then $T \in \mathfrak{S}_2$ if and only if*

$$\int |K(x, y)|^2 d\mu(x) d\mu(y) < \infty.$$

In this case,

$$||T||_2 = \left(\int |K(x, y)|^2 d\mu(x) d\mu(y) \right)^{1/2},$$

(see Kato, 1984; Weidmann, 1980). The following theorem was proved in Kato (1965) is the basic to the theory of differential operators where the perturbation is an operator of multiplication.

Theorem 2.4. *Suppose that*

$$(H - z)^{-k} - (H_0 - z)^{-k} \in \mathfrak{S}_1, \tag{2.1}$$

for some $k = 1, 2, \ldots$ and all z with $\operatorname{Im} z \neq 0$. *Then the wave operator $\mathcal{W}_\pm(H_0, H)$ exist and complete.*

3. SCHRÖDINGER OPERATOR WITH MAGNETIC VECTOR POTENTIAL

In the present section, formal Schrödinger operators are considered

$$\hat{H}_0 = -\sum_{k,j}^{d}(D_j - ia_j)^2 + q$$

with real-valued functions $\mathfrak{a}$ and q such that

$$\mathfrak{a} = (a_1, a_2 \ldots , a_d) \in [L_{2,loc}(\Re^d)]^d, \quad 0 \le q \in L_{1,loc}(\Re^d) \tag{3.1}$$

where $[L_2(\Re^d)]^d$ be the cartesian product of $L_2(\Re^d)$ with inner product:

$$\langle \mathfrak{u}, \mathfrak{v} \rangle = \sum_{j=1}^{d} \langle u_j, v_j \rangle, \quad \| \mathfrak{u} \| = \sqrt{\langle \mathfrak{u}, \mathfrak{u} \rangle}.$$

For $1 \le j \le d$ let $\partial_j = \frac{\partial}{\partial x_j}$, be the j-th partial derivative, each acting on the space of distributions $\mathcal{D}'(\Re^d)$ on $\Re^d$ i.e. $\partial_j : C_0^\infty(\Re^d) \longrightarrow \mathcal{D}'(\Re^d)$. Take $Du = \nabla u - i\mathfrak{a}u \in [\mathcal{D}'(\Re^d)]^d$ where $\nabla = (\partial_1, \partial_2, \ldots , \partial_d)$ and $u \in L_{2,loc}(\Re^d)$ with $i = \sqrt{-1}$. This operator is associated to the sesquilinear $\mathfrak{h}(u, v)$ which defined as follows:

$$\begin{aligned}
\mathcal{Q}(\mathfrak{h}) &= \{u \in L_2(\Re^d) : Du \in [L_2(\Re^d)]^d, \quad q^{1/2}u \in L_2(\Re^d)\} \\
\mathfrak{h}(u, v) &= \langle Du, Dv \rangle + \langle q^{1/2}u, q^{1/2}v \rangle \\
&= \sum_{j=1}^{d} \langle \partial_j u - ia_j u, \partial_j v - ia_j v \rangle + \langle q^{1/2}u, q^{1/2}v \rangle \\
&= \sum_{j=1}^{d} \langle D_j u, D_j v \rangle + \langle q^{1/2}u, q^{1/2}v \rangle,
\end{aligned}$$

with norm:

$$|||u|||_{\mathfrak{h}}^2 = |Du|_{L_2}^2 + \left|q^{1/2}u\right|_{L_2}^2 + |u|_{L_2}^2.$$

Clearly $C_0^\infty(\Re^d) \subset \mathcal{Q}(\mathfrak{h})$. It is also known that, $C_0^\infty(\Re^d)$ is dense with respect to $|||u||| = [\mathfrak{h}(u, u) + \langle u, u \rangle]^{1/2}$ (see Biver-Weinholtz, 1982a,b,c; Leinfelder and Siameder, 1981).

Lemma 3.1. *The sesquilinear $\mathfrak{h}(u, v)$ is symmetric positive closed form with its norm, hence there exists a unique self-adjoint operator H_0 satisfying*

$$\begin{aligned}
\mathfrak{D}(\hat{H}_0) &= \{u \in \mathcal{Q}(\mathfrak{h}) : \mathfrak{h}(u, \cdot) \in L_2(\Re^d)\}, \\
\mathfrak{h}(u, v) &= \langle \hat{H}_0 u, v \rangle, \quad u \in \mathfrak{D}(\hat{H}_0), \quad v \in \mathcal{Q}(\mathfrak{h})
\end{aligned}$$

Proof: Since

$$\overline{\mathfrak{h}(u,v)} = \overline{\langle Du, Dv\rangle} + \overline{\langle q^{1/2}u, q^{1/2}v\rangle} = \langle Dv, Du\rangle + \langle q^{1/2}v, q^{1/2}u\rangle = \mathfrak{h}(v,u),$$

hence $\mathfrak{h}(u,v)$ is positive and symmetric. We know that $\mathfrak{h}(u,v)$ is closed if and only if $(\mathcal{Q}(\mathfrak{h}), ||| \cdot |||)$ is complete by Kato (1984, Theorem (1.11) chapter (IV)). Suppose (u_n) is a Cauchy sequence in $\mathcal{Q}(\mathfrak{h})$ i.e.

$$|||\, u_n - u_l \,|||_{\mathfrak{h}} = \|D(u_n - u_l)\| + \|q^{1/2}(u_n - u_l)\| + \|u_n - u_l\| \longrightarrow 0,$$

as $n, l \longrightarrow \infty$ and hence $\|u_n - u||_{L_2(\Re^d)} \longrightarrow 0$ as $n \longrightarrow \infty$. Also, $\|q^{1/2}u_n - q^{1/2}u\| = \|q^{1/2}(u_n - u)\| \longrightarrow 0$ as $n \longrightarrow \infty$, then $q^{1/2}u_n \longrightarrow q^{1/2}u$ and $q^{1/2}u \in L_2(\Re^d)$.

Since the derivative is continuous , $\partial_j u_n \longrightarrow \partial_j u \in L_2(\Re^d)$. Finally, a_j, u_n and $u \in L_2(\Re^d)$ then, $a_j u_n$, $a_j u \in L_2(\Re^d)$ and $\|a_j u_n - a_j u\| = \|a_j(u_n - u)\| \longrightarrow 0$ as $n \longrightarrow \infty$. Consequently, $\|D_j u_n - \mathbb{D}_j u\| = \|(\partial_j u_n - a_j u_n) - (\partial_j u - a_j u)\| \leq \|\partial_j u_n - \partial_j u\| + \|a_j u_n - a_j u\| \longrightarrow 0$, as $n \longrightarrow \infty$. Then, $\mathbb{D}_j u_n \longrightarrow \mathbb{D}_j u$ and $u \in \mathcal{Q}(\mathfrak{h})$. Thus $\mathcal{Q}(\mathfrak{h})$ is symmetric positive closed form.

Applying the *Fridrich*'s Theorem, there exists a unique non-negative self-adjoint operator H_0 associated with $\mathcal{Q}(\mathfrak{h})$ such that

$$\mathfrak{D}(\hat{H}_0) = \{u \in \mathcal{Q}(\mathfrak{h}) : \mathfrak{h}(u, \cdot) \in L_2(\Re^d)\}$$

$$\mathfrak{h}(u,v) = \langle H_0 u, v\rangle, \quad \text{for} \quad u \in \mathfrak{D}(\hat{H}_0), v \in u \in \mathcal{Q}(\mathfrak{h}).$$

Lemma 3.2. *Let $\lambda > 0$, $u \in L_2(\Re^d)$ and assume the condition (3.1). Then*

$$|\, (\hat{H}_0 + \lambda)^{-1} u \,| \leq (-\Delta + \lambda)^{-1} \,|\, u \,| \,.$$

For the proof of this Lemma see Leinfelder and Siameder (1981).
Suppose that

$$\begin{cases} \mathfrak{a} \in [L_{4,loc}(\Re^d)]^d, \quad \operatorname{div} \mathfrak{a} = \sum_{j=1}^d \partial_j a_j \in L_{2,loc}(\Re^d) \\ 0 \leq q \in L_{2,loc}(\Re^d). \end{cases} \tag{3.2}$$

Observe, that the condition (3.2) implies (3.1), so that all the above results are valid. Also one have the following theorem (for details see Leinfelder and Siameder, 1981).

Theorem 3.3. *Assume (3.2) and let $\hat{H}_0$ be the self-adjoint operator in Lemma (3.1). Then $C_0^\infty(\Re^d)$ is an operator core of $\hat{H}_0$, i.e. $\hat{H}_{0|_{C_0^\infty(\Re^d)}} = -(\nabla - i\mathfrak{a})^2 + q$ is essentially self-adjoint. In addition, if $q_1 = \min(q, 0)$ is $\Delta -$ bounded* *with relative bound $\gamma < 1$, then $\hat{H}_{0|_{C_0^\infty(\Re^d)}} = -(\nabla - i\mathfrak{a})^2 + q$ is essentially self-adjoint and semi-bounded from below.*

Theorem 3.4. *Suppose $\mathfrak{a} \in [L_{4,loc}(\Re^d)]^d$, $\operatorname{div}\mathfrak{a} \in L_{2,loc}(\Re^d)$ and assume $q = q_1 + q_2$ with $q_1, q_2 \in L_{2,loc}(\Re^d)$, $q_2 \le 0$. Let q_2 be $\Delta -$ bounded with relative bound $\gamma < 1$ and suppose q_1 satisfies $q_1 \ge -c|x|^2$ with a constant $c > 0$. Then $\hat{H}_{0|_{C_0^\infty(\Re^d)}} = -(\nabla - i\mathfrak{a})^2 + q$ is essentially self-adjoint.*

Lemma 3.5. *Assume that $q = q_1 + q_2 = V_0 + V$, where V_0, V are the multiplication operator by the functions v_0, v such that:*

$$V_0 = \overline{V_0} \in L_\infty(\Re^d), \quad (V_0 f)(x) = v_0(x) f(x), \tag{3.3}$$

$$(Vf)(x) = v(x) f(x), \quad V = \overline{V}, \tag{3.4}$$

$$|v(x)| \le C(1 + |x|)^{-\rho}, \quad \rho > d. \tag{3.5}$$

The condition (3.5) is called the short range potential. Define the operators

$$\hat{H}_0 = -(\nabla - i\mathfrak{a})^2 + V_0, \quad \hat{H} = \hat{H}_0 + V$$

with $\mathfrak{D}(\hat{H}) = \mathfrak{D}(\hat{H}_0) = C_0^\infty(\Re^d)$. Then wave operators $\mathcal{W}_\pm(\hat{H}, \hat{H}_0)$ exist and are complete.

Proof: By Theorems (3.3) and (3.4) the operators $\hat{H}_0$ and $\hat{H}$ are essentially self-adjoint. Also, from Lemma (4.2) we get

$$| V |^{1/2} (\hat{H}_0 + \lambda)^{-1} u \le | V |^{1/2} | (\hat{H}_0 + \lambda)^{-1} u | \le | V |^{1/2} (-\Delta + \lambda)^{-1} | u |,$$

hence the operator $|V|^{1/2}(\hat{H}_0 - \lambda)^{-1}$ is an integral operator which belongs to $\mathfrak{S}_2$. □

4. THE OPERATOR OF HIGHER ORDER

Throughout this section denote by D_j the operator $(1/i)(\partial/\partial x_j)$, and we use the standard multi-index notation: if $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, is an $n-$tuple of nonnegative integers,

$$x^\alpha = (x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_d^{\alpha_d}) \quad \text{and}$$

$$D^\alpha = (-1)^{|\alpha|} \prod_{j=1}^{m} \frac{\partial^{\alpha_j}}{\partial x^{\alpha_j}} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}},$$

where $|\alpha| = \sum_{j=1}^{d} \alpha_j$.

Let $\mathfrak{t}_{00}$ is sesquilinear form with domain $D(\mathfrak{t}_{00}) = W^{2,m}(\Re^d)$ and

$$\mathfrak{t}_{00}(u, v) = \sum_{|\alpha|,|\beta|\leq m} \langle b_{\alpha\beta}(x) D^\beta u, D^\alpha v\rangle_{L^2},$$

where m is non-negative integer, $m \leq d$, β and α are multi-indices and $b_{\alpha\beta}(x)$ satisfies the following :

(I) $b_{\alpha\beta}$ are bounded real measurable functions on $\Re^d$ and all $b_{\alpha\beta}$ with $|\alpha| = |\beta|$ are uniformly continuous on $\Re^d$,

(II) $b_{\alpha\beta} = b_{\beta\alpha}$,

(III) $\exists\ C_0 > 0$ such that $\sum_{|\alpha|=|\beta|=m} b_{\alpha\beta}\xi^{\alpha+\beta} \geq C_0|\xi|^{2m}$, ξ, $x \in \Re^d$. (uniform strong ellipticity).

(IV) $\exists\ C_1 > 0$ such that $\sum_{|\alpha|=|\beta|=m}\{b_{\alpha\beta} + v_0 + v(x)\}\xi^{\alpha+\beta} \geq C_1|\xi|^{2m}$, $\xi, x \in \Re^d$. The principle part of t_{00} will be denoted by:

$$\mathfrak{s}_{00}(u, v) = \sum_{|\alpha|=|\beta|=m} \langle b_{\alpha\beta}(x) D^\beta u, D^\alpha v\rangle,$$

$u, v \in W^{2,m}(\Re^d)$. Note that each of t_{00} and $\mathfrak{s}_{00}$ are bounded forms. We shall prove that t_{00} is positive form. For this we need the following:

Lemma 4.1. *Assume that $\mathfrak{s}_{00}$ has constant coefficient, then for all $u \in W^{2,m}(\Re^d)$*

$$\mathfrak{s}_{00}(u) \geq \nu C_0\|u\|_m^2,$$

where ν depend on d, m.

Proof: By using the Fourier transform of u, then

$$\int_{\Re^d} D^\alpha u D^\beta u\, dx = \int_{\Re^d} \hat{D}^\alpha u \hat{D}^\beta u\, dx = \int_{\Re^d} x^{\alpha+\beta}|\hat{u}|^2 dx.$$

Thus

$$\mathfrak{s}_{00}(u) = \sum_{|\alpha|=|\beta|=m} b_{\alpha\beta}\int_{\Re^d} x^{\alpha+\beta}|\hat{u}|^2 dx.$$

By the ellipticity assumption (III) we get

$$\mathfrak{s}_{00}(u) \geq \nu C_0 \int_{\Re^d} x^{2m}|\hat{u}(x)|^2 dx, \qquad \nu(r, d).$$

By Parseval's identity we obtain that

$$\mathfrak{s}_{00}(u) \geq \nu C_0\left[\int_{\Re^d}\sum_{|\alpha|=m}|D^\alpha u|^2 dx\right]^{1/2} \geq \nu C_0\|u\|_m^2, \qquad \nu(m, d).$$

$\square$

Theorem 4.2. *If $\mathfrak{s}_{00}$ has constant coefficients. Then for all $u \in W^{2,m}(\Re^d)$ and some constant $\varrho > 0$*

$$\mathfrak{t}_{00}(u) \geq \varrho \|u\|^2.$$

Proof: We have

$$\mathfrak{t}_{00}(u) = \mathfrak{s}_{00} + \sum_{|\alpha| \leq m-1, |\beta| \leq m-1} \int_{\Re^d} b_{\alpha\beta}(x) D^{\alpha}(u) D^{\beta}(u) dx.$$

By the Cauchy-Schwartz inequality we get

$$\begin{aligned} \left| \int_{\Re^d} b_{\alpha\beta}(x) D^{\alpha} u D^{\beta} u dx \right| &\leq \sup_{x \in \Re^d} |b_{\alpha\beta}(x)| \left[\int_{\Re^d} |D^{\alpha} u|^2 dx \right]^{1/2} \left[\int_{\Re^d} |D^{\beta} u|^2 dx \right]^{1/2} \\ &\leq k \left[\int_{\Re^d} |D^{\alpha} u|^2 dx \right]^{1/2} \left[\int_{\Re^d} |D^{\beta} u|^2 dx \right]^{1/2}. \end{aligned}$$

Hence

$$\mathfrak{t}_{00}(u) \geq \mathfrak{s}_{00} - \left[\sum_{|\alpha| \leq m-1} \int_{\Re^d} |D^{\alpha} u|^2 dx \right]^{1/2} \left[\sum_{|\beta| \leq m-1} \int_{\Re^d} |D^{\beta} u|^2 dx \right]^{1/2}.$$

By Lemma (2.1) we get

$$\begin{aligned} \mathfrak{t}_{00}(u) &\geq \nu C_o \left[\int_{\Re^d} \sum_{|\alpha|=m} |D^{\alpha} u|^2 dx \right]^{1/2} - k\|u\|_{m-1}^2 \\ &= \nu C_0 \left[\int_{\Re^d} \left\{ \sum_{|\alpha|=m} |D^{\alpha} u|^2 - \sum_{|\alpha|<m} |D^{\alpha} u|^2 + \sum_{|\alpha|<m} |D^{\alpha} u|^2 \right\} dx \right]^{1/2} \\ &\quad - k\|u\|_{m-1}^2. \end{aligned}$$

Hence

$$\begin{aligned} \mathfrak{t}_{00}(u) \geq \nu C_0 \left[\int_{\Re^d} \sum_{|\alpha| \leq m} |D^{\alpha} u|^2 dx \right]^{1/2} - \nu C_o \left[\int_{\Re^d} \sum_{|\alpha|<m} |D^{\alpha} u|^2 dx \right]^{1/2} \\ - k\|u\|_{m-1}^2, \end{aligned}$$

i.e.,

$$\mathfrak{t}_{00}(u) \geq \nu C_0 \|u\|_m^2 - (\nu C_0 + k)\|u\|_{m-1}^2,$$

thus there exists a constant $\varrho(\nu, C_0, k)$ such that $\mathfrak{t}_{00}(u) \geq \|u\|_m^2 \geq \varrho\|u\|^2$. □

Theorem 4.3. *Assume that* $\mathfrak{t}_{00} = \mathfrak{s}_{00}$ *that the coefficients* $b_{\alpha\beta}(x)$ *are uniformly continuous,* $|\alpha| = |\beta| = m$. *Then, for all* $u \in W^{2,m}(\Re^d)$,

$$\mathfrak{t}_{00}(u) \geq \nu C_0 \|u\|^2,$$

Proof: Since the coefficients $b_{\alpha\beta}(x)$ are uniformly continuous, so for every $\varepsilon > 0$, there exists a positive number δ such that

$$|b_{\alpha\beta}(x) - b_{\alpha\beta}(y)| < 0 \quad \text{if} \quad |x - y| < \delta, \quad x, y \in \Re^d.$$

Let $u \in W^{2,m}(\Re^d)$ and the diameter of supp.(u) is less than δ, $x_0 \in$ supp. (u), then

$$\mathfrak{t}_{00_{x_0}}(u) = \sum_{|\alpha|=\beta=m} \int_{supp.(u)} b_{\alpha\beta}(x_0) D^\alpha u D^\beta u \, dx.$$

Hence

$$\begin{aligned} t_{00}(u) &= \mathfrak{t}_{00_{x_0}}(u) + \sum_{|\alpha|=\beta=m} \int_{supp.(u)} [b_{\alpha\beta}(x) - b_{\alpha\beta}(x_0)] D^\alpha u D^\beta u \, dx \\ &\geq \mathfrak{t}_{00_{x_0}}(u) - \varepsilon \left[\int_{\Re^d} \sum_{|\alpha|=m} |D^\alpha u|^2 dx \right]^{1/2} \left[\int_{\Re^d} \sum_{|\beta|=m} |D^\beta u|^2 dx \right]^{1/2} \\ &\geq \mathfrak{t}_{00_{x_0}}(u) - k\varepsilon \|u\|_m^2. \end{aligned}$$

Apply Theorem(2.2) to the form $\mathfrak{t}_{00_{x_0}}(u)$, which has constant coefficients we get

$$\mathfrak{t}_{00}(u) \geq \varrho \|u\|_m^2 - k\varepsilon \|u\|_m^2 = (\varrho - k\varepsilon)\|u\|_m^2 \geq (\varrho - k\varepsilon)\|u\|^2 \qquad \square$$

One can generalize the above theorem if the condition $\mathfrak{t}_{00} = \mathfrak{s}_{00}$ is omited in the following:

Corollary 4.4. *The above Theorem remains true without the assumption* $\mathfrak{t}_{00}(u) = \mathfrak{s}_{00}$.

Proof: Assume $u \in W^{2,m}(\Re^d)$ and $supp.(u)$ has a diameter less than δ, where δ is a constant guaranteed for the form $\mathfrak{s}_{00}$ of the Theorem (2.4), then

$$\mathfrak{t}_{00}(u) = \mathfrak{s}_{00} + \sum_{|\alpha| \leq m-1, |\beta| \leq m-1} \int_{supp.(u)} b_{\alpha\beta}(x) D^\alpha u D^\beta u \, dx.$$

By Lemma (2.1) and Cauchy–Schwartz inequality we get

$$\mathfrak{t}_{00}(u) \geq \varrho C_0 \|u\|_m^2 - k\|u\|_{m,supp.(u)} \|u\|_{m-1,supp.(u)}.$$

Using the inequality $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2, \quad a, b \geq 0$, so the above inequality becomes

$$\mathfrak{t}_{00}(u) \geq \lambda C_0 \|u\|_m^2 - k\varepsilon \|u\|_{m,supp.(u)} - k\varepsilon^{-1} \|u\|_{m-1,supp.(u)}.$$

Using Agmon (1975) Lemma (7.3), one have

$$\|u\|_{m-1,supp.(u)} \leq \nu\delta \|u\|_{m,supp.(u)} = \nu\delta \|u\|_m,$$

it follows that $\mathfrak{t}_{00}(u) \geq (\lambda C_0 - k\varepsilon - k\varepsilon^{-1}\delta^2)\|u\|_m^2$, $\lambda(C_0, k, \varepsilon, \delta)$, i.e. $\mathfrak{t}_{00}$ is non-negative form. □

Take $b_{\alpha\beta}$ are constant then $\mathfrak{t}_{00}(u, v)$ is non-negative symmetric closed form with $W^{2,m}(\Re^d)$. Let H_{00} be the non-negative self-adjoint operator associated with $\mathfrak{t}_{00}(u, v)$ in the sense of Friedrichs (see Kato, 1984). Namely, H_{00} is the unique non-negative self-adjoint operator such that:

$$\mathfrak{t}_{00}(u, v) = \langle H_{00}u, v\rangle, \quad u \in \mathfrak{D}(H_{00}), v \in \mathfrak{D}(\mathfrak{t}_{00}),$$

$$H_{00} = \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\alpha|} D^\alpha (b_{\alpha\beta} D^\beta). \tag{4.3}$$

Let P be the polynomial given by:

$$P(x) = \sum_{|\alpha|,|\beta|\leq m} b_{\alpha\beta} x^{\alpha+\beta},$$

$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, + \cdots, \alpha_d + \beta_d)$. It is given well-known that:

$$\mathfrak{D}(H_{00}) = W^{2,2m}(\Re^d) \quad H_{00}u = P(D)u, \qquad u \in W^{2,2m}(\Re^d).$$

The spectrum $\sigma(H_{00})$ of H_{00} is equal to $[\lambda_{min}, \infty[$ where $\lambda_{min} = \inf_{x\in\Re^d} P(x)$. We use the following definition (see Agmon, 1970; Kuroda, 1973).

Definition 4.5. $\lambda \in \Re$ is said to be critical value of the polynomial P if there exists $x \in \Re^d$ such that $P(x) = \lambda$ and $grad P(x) = 0$.

The set e_0 of all critical values of P is closed set of measure zero and H_{00} is absolutely continuous in $\Re - e_0$.

Consider the sesquilinear form

$$\mathfrak{t}_0(u, v) = \mathfrak{t}_{00}(u, v) + \langle V_0 u, v\rangle, \quad u, v \in \mathfrak{D}(t_0) = W^{2,m}(\Re^d),$$

where V_0 is the multiplication operator satisfies (3.3). Thus, t_0 is closed non-negative symmetric form. So there exists a unique non-negative self-adjoint operator H_0 associated with $\mathfrak{t}_0$ in the sense of *Friedrichs*. Consequently, H_0 is the realization of the formal differential operator

$$H_0 = \sum_{|\alpha|,|\beta|\le m} (-1)^{|\alpha|} D^\alpha (b_{\alpha\beta} D^\beta) + V_0.$$

By the same way one can define the operator H by the sesquilinear form

$$\mathfrak{t}(u, v) = \mathfrak{t}_0(u, v) + \langle Vu, v\rangle, \quad u, v \in \mathfrak{D}(\mathfrak{t}) = W^{2,m}(\Re^d),$$

where V is the multiplication operator satisfies (3.4) and (3.5). One have H is the realization of the differential operator $H_0 + V$,

$$H = \sum_{|\alpha|,|\beta|\le m} (-1)^{|\alpha|} D^\alpha (b_{\alpha\beta} D^\beta) + V_0 + V,$$

which is non-negative self-adjoint operator associated with $\mathfrak{t}$. The main theorem is the following:

Theorem 4.6. *Let H_{00}, H_0 and H be a self-adjoint operators in $L_2(\mathbb{R}^d)$ connected by the equalities (4.1), (4.2) and (4.3) with*

$$\mathfrak{D}(H) = \mathfrak{D}(H_0) = \mathfrak{D}(H_{00}).$$

Then the wave operator $\mathcal{W}_\pm(H, H_0)$ exist and are complete with condition (3.5).

Proof: To prove the existence and the completeness of the wave operator $\mathcal{W}_\pm(H_0, H)$ it is sufficient, for $k = 1$, to prove (2.1). By the resolvent equation and (4.4), Eq. (2.1) can be rewritten as follows:

$$\begin{aligned}(H - z)^{-1} - (H_0 - z)^{-1} &= -(H - z)^{-1} V (H_0 - z)^{-1} \\ &= (H - z)^{-1}(H_{00} - z)(H_{00} - z)^{-1} V (H_{00} - z)^{-1}(H_{00} - z)(H_0 - z)^{-1}.\end{aligned}$$

From the conditions on V_0, V and (IV) the operator $(H - z)^{-1}(H_{00} - z) = I - (V_0 + V)(H - z)^{-1} = [I - (V_0 + V)(H - \mu)^{-1}]$ is bounded. Also the operators $(H_{00} - z)^{-1}(H_0 - z) = I - V_0(H_0 - z)^{-1}$ bounded. It suffices to check that:

$$(H_{00} - z)^{-1} V (H_{00} - z)^{-1} = ((H_{00} - z)^{-1}|V|^{1/2}) \operatorname{sign} V (|V|^{1/2}(H_{00} - z)^{-1}) \in \mathfrak{S}_1.$$

Because of the product of tow *Hilbert–Schmidt* operators is trace class operators see Lemma (2.1), it is necessary to show only that

$$(H_{00} - z)^{-1}|V|^{1/2} \in \mathfrak{S}_2, \qquad |V|^{1/2}(H_{00} - z)^{-1} \in \mathfrak{S}_2.$$

Let $\mathcal{F} : L_2(\Re^d) \longrightarrow L_2(\Xi^d)$ be the Fourier transformation and Ξ^d the duel space of $\Re^d$, i.e., $\mathcal{F}$ is the integral operator with kernel $(2\pi)^{-d/2} \exp(-i < x, \xi >)$. From the properties of $\mathcal{F}$ and the ellipticity of the non-negative selfadjoint H_{00} one have, for $\mu \in \rho(H_{00})$, $\mu < 0$, the resolvent operator $(H_{00} - \mu)^{-1}$ is bounded and

$$\mathcal{F}(H_{00} - \mu)^{-1} = (P(x) - \mu)^{-1}\mathcal{F} = \left(\sum_{|\alpha|,|\beta| \leq m} b_{\alpha\beta} x^{\alpha+\beta} - \mu \right)^{-1} \mathcal{F}$$
$$\leq \left(\sum_{|\alpha|,|\beta| = m} b_{\alpha\beta} x^{\alpha+\beta} - \mu \right)^{-1} \mathcal{F} \leq C_2(|\ x\ |^{2m} - \mu)^{-1}\mathcal{F}.$$

Hence

$$\mathcal{F}(H_{00} - \mu)^{-1}|V|^{1/2} f(x) \leq (|\ x\ |^{2m} - \mu)^{-1}\mathcal{F}|V|^{1/2} f(x)$$
$$= (2\pi)^{-d/2}(|\ x\ |^{2m} - \mu)^{-1} \int e^{-ixy}|V(y)|^{1/2} f(y)dy,$$

$(H_{00} - \mu)^{-1}|V|^{1/2}$ is an integral operator with kernel less than or equal to

$$(2\pi)^{-d/2} C_2 \mid v(x) \mid^{1/2} \exp^{-ix.y}(|\ x\ |^{2m} - \mu)^{-1} \in L^2(\Re^{2d}).$$

Using the proposition (2.3) or (2.4) one get the operator $|\ V\ |^{1/2}\ (H_{00} - \mu)^{-1} \in \mathfrak{S}_2$, if $\rho > d$. By the virtu of the adjoint operator in *Hilbert-Schmidt* class is *Hilbert-Schmidt*, one have the operator $|V|^{1/2}(H_0 - \mu)^{-1} \in \mathfrak{S}_2$, and hence (4.5) is holds. □

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